

Discount Rates for Public Investment in Closed and Open Economies

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I. INTRODUCTION

Let us imagine a perfectly competitive economy operating under the usual neo-classical assumptions of decreasing returns, convexity of preferences, etc. To make our discussion simple we further limit ourselves to a one-good two-period framework, in which initial resources can be allocated to immediate consumption or to investment, and in which investment produces the goods available for future consumption. With a perfect loan market consumers will equate their marginal rates of time preference to the rate of interest, while private firms will carry investment to the point where the marginal productivity of private capital is equal to the interest rate. Suppose now that— for reasons that we shall take as given—part of the total investment of the economy takes place in publicly-owned "firms". It is then obvious that the optimal amount of public investment is found by equating the marginal productivity of public investment to the market rate of interest; the marginal conditions for efficiency or Pareto-optimality will then be satisfied all around.

Suppose now that there are distortions in the economy, e.g. in the form of taxes, which prevent the equalization of marginal rates of substitution and transformation in the private sector. Then it is not any longer so easy to identify the correct rate of discount to use in the public sector. In a recent paper, Baumol (1)¹ has argued that, under certain assumptions, the corporate income tax drives a wedge between the marginal rate of time preference of consumers and the marginal rate of transformation in private firms. Baumol concludes that in such a case "there remains an inescapable indeterminacy in the choice of a discount rate on government projects", since this discount rate cannot simultaneously be equal to the marginal rate of time preference of consumers and the marginal productivity of private capital. Baumol further remarks that he can find no theoretical grounds for accepting

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² References in square brackets are listed on pp. 411-12, below.

either of these rates (or presumably any third alternative) as the appropriate discount rate to use in the public sector, and that any choice of such a rate will to some extent be arbitrary.

Our objective in this paper is to determine the rate of discount for public investment which is optimal on efficiency grounds, given that there exists a wedge between the marginal rate of time preference and the marginal productivity of private capital (due, for instance, to the corporation income tax). This is an exercise in the theory of second-best welfare economics. We shall first analyze the problem for a closed economy. Then we shall extend the discussion to an open economy to allow for foreign lending and foreign direct investment. In Section VII we provide some numerical illustrations of our main conclusions.

Some of our results—especially in Section IV—bear a strong resemblance to those presented by Ramsey [6] and Usher [9] in comments on the Baumol article.¹ By using a streamlined general equilibrium model of the economy we provide a theoretical foundation for their conclusions and sharpen them considerably. Our approach also has a certain similarity to that of Diamond [3].

For followers of the debate on the "social rate of discount" it may perhaps be worthwhile to stress that there are many aspects of it which we shall not touch upon in this contribution. Some writers have maintained, for example, that the time preferences of present-day consumers do not reflect adequately the time preference concept appropriate for society, and that the evaluation of public investment should attempt to correct for this externality; e.g. see Marglin [5]. If output from the public investment project is in the nature of a public good, we face of course the problem of optimal allocation of resources to public goods production and the difficulties of implementing the Samuelson conditions [8]. Neither of these problems arises in our models. We also neglect the problems raised by uncertainty, which would justify a separate paper.

II. A SIMPLE CLOSED ECONOMY

We begin by describing a simple one-good economy in a two-period context. In the main text of the paper we make the further simplifying assumption that consumption and investment decisions in the private sector can be analysed as if there were only one consumer and one private firm in the economy. In the Appendix we show that the general case with many consumers and firms leads to exactly the same results. It should be stressed, therefore, that the results in the text itself are based on the Pareto-optimality criterion; no "social" evaluation is involved which is not based on individual preference orderings. We can make the same point by referring to Samuelson's classic article [7]

¹ A subsequent contribution by Baumol [2] takes account of Ramsey's criticism and revises the view expressed in [1].

on social indifference curves, in which it is shown that the aggregate preferences of the economy can be represented by a social utility function when optimal re-allocations of income are taking place; this is indeed what happens in the n -consumer model in the Appendix, and explains why the two formulations are equivalent.

In the initial period the total resources of the economy are given and equal to w . This can be used for immediate consumption (c_1), private investment (y) and public investment (z), so that for the economy as a whole

$$(1) \quad c_1 + y + z = w.$$

The outputs of the private and the public sectors constitute the resources available for future consumption; in our simple framework, total second-period output is simply equal to the sum of outputs in the two sectors.¹ Thus, letting $f(y)$ be the production function for the private sector and $g(z)$ that for the public sector, we have

$$(2) \quad c_2 = f(y) + g(z).$$

We assume the first derivatives of these functions to be positive and the second derivatives negative, at least locally.

Finally, we assume that consumer preferences can be represented by a social utility function with present and future consumption as arguments; thus we have

$$(3) \quad U = U(c_1, c_2).$$

This function will be assumed to have the usual concavity properties familiar from demand analysis.

The reader may easily convince himself that an optimum for this economy can be characterized by the following equations:

$$(4) \quad f'(y) = g'(z) = U_1/U_2.$$

Here U_i is the partial derivative of U with respect to c_i ($i=1, 2$). At an optimum, the marginal productivities of private and public capital must be the same and equal to the marginal rate of time preference.

We now describe the equilibrium of the private sector. The initial resources of the economy are owned by consumers, who can use them for three purposes: initial consumption, private investment, or lending to the government. Letting b be the amount lent to the public sector, we thus have the budget constraint of private consumers:

$$(5) \quad c_1 + y + b = w.$$

¹ This feature of the model, which may seem slightly artificial, could easily be amended by assuming the two sectors to produce different commodities. We adhere to this stylized formulation throughout the paper, in particular because we wish the concept of "marginal rate of time preference" to have a precise meaning within our model.

As seen by consumers, there are three components in the amount available for future consumption. One is the return from private investment, net of taxes. There is a proportional tax rate on profits, defined as $f(y) - y$; thus, interest on capital is not deductible for tax purposes. This corresponds to the usual treatment of equity capital. Future income from investment then becomes $f(y) - t(f(y) - y)$. In addition, what is lent to the government in the first period is paid back in the second with interest added at the rate r . Finally, consumers are paid a lump-sum subsidy in the amount a , which is equal to the surplus earned in government production. We then have that

$$(6) \quad c_2 = f(y) - t(f(y) - y) + b(1+r) + a.$$

Substituting from (5) we get

$$(7) \quad c_2 = f(y) - t(f(y) - y) + (w - c_1 - y)(1+r) + a.$$

Substituting into the utility function (3) and maximizing with respect to c_1 and y , we obtain the first-order conditions:

$$(8) \quad U_{c_1}/U_s = 1 + r,$$

$$(9) \quad f'(y) = 1 + r/(1 - t).$$

These two equations give us the demand functions for present consumption and private investment as functions of r , t , a and w ; residually, they also determine the demand for government bonds or, equivalently, the supply of funds to the government.

The system determining private sector equilibrium is closed by using the resources constraint (1); taken together with (5), this can be written alternatively as

$$(10) \quad z = b.$$

We now have three equations to determine the equilibrium of the private sector with respect to c_1 , y and r .

The public sector, having fixed the value of r , determines a and z so as to achieve a constrained optimum for the economy. The values of a and z must be chosen so as to satisfy the budget equation of the public sector in the second period, viz.

$$(11) \quad g(z) + t(f(y) - y) = (1+r)b + a.$$

Finally, the level of public investment must be determined by an equation of the form

$$(12) \quad g'(z) = 1 + p.$$

where p is the discount rate for public investment decisions which is optimal from an efficiency point of view. This is in the nature of a shadow price, the correct determination of which is the main concern of this article.

The simplest case is, of course, $t=0$. Then the marginal productivity of private capital is equal to the marginal rate of time preference, and the optimality conditions (4) are satisfied when $p=r$. This is a "first best" solution. However, when $t>0$, the efficiency condition is not satisfied in the private sector, and second-best theory tells us that the condition $g'(z) = 1+r$ has no longer any claim to being considered a true optimality condition. A more detailed analysis is therefore required.¹

III. SOME COMPARATIVE STATICS

As a preliminary to this analysis we need some comparative statics results for the private sector. In particular, we are interested in the interest derivatives of private investment and consumption. Differentiating (9) with respect to r we obtain

$$(13) \quad \partial y/\partial r = 1/(1-t)f''(y).$$

At the optimum, marginal productivity must be decreasing; this derivative is therefore negative.

The derivative of present consumption with respect to initial wealth is [from (8)]:

$$(14) \quad \partial c_1/\partial w = 1/D \{ -(1+r)U_{12} + (1+r)^2 U_{22} \},$$

where the second-order maximum condition implies that

$$D = U_{11} - 2(1+r)U_{12} + (1+r)^2 U_{22} < 0.$$

The sign of the expression in braces cannot be determined *a priori*, but it is reasonable, in the light of empirical observation, to assume that it is positive.

Differentiating in (8) with respect to r , we obtain

$$\partial c_1/\partial r = 1/D \{ (w - c_1 - y) \{ -U_{12} + (1+r)U_{22} \} + (1/D)U_s \}.$$

Substituting from (14) we then get

$$(15) \quad \frac{\partial c_1}{\partial r} = \frac{1}{1+r} (w - c_1 - y) \frac{\partial c_1}{\partial w} + \frac{1}{D} U_s.$$

The first term is the income effect, which is positive under our assumptions,² and the second term is the substitution effect, which is negative.

It is also of interest to evaluate the derivative $\partial c_1/\partial w$. Differentiating in (8) and substituting from (14) one obtains easily that

¹ In the general case where a fraction k of the interest bill is deductible for tax purposes, condition (9) becomes $f'(y) = 1 + r(1 - kr)/(1 - t)$.

² Note that the coefficient of the income derivative is proportional to $(w - c_1 - y)$ and not to $(w - c_1)$, which is total saving. Since, for given y , the returns on investment do not vary with r , there is no income effect on investment.

$$(16) \quad \frac{\partial c_1}{\partial a} = \frac{1}{1+r} \cdot \frac{\partial c_1}{\partial w}$$

IV. THE OPTIMAL DISCOUNT RATE

We are now ready to tackle the problem of the optimal discount rate for public investment. Substituting from (7) into (3) we can write the utility of consumers as

$$U = U(c_1, f(y) - i(f(y) - y) + (w - c_1 - y)(1+r) + a).$$

The government chooses z and a so as to maximize this function subject to its budget constraint (11); after substitution from (10) and (1) we can write this as

$$g(w - c_1 - y) + i(f(y) - y) = (1+r)(w - c_1 - y) + a.$$

The function to be maximized with respect to z and a is thus the Lagrangian

$$L = U(c_1, f(y) - i(f(y) - y) + (w - c_1 - y)(1+r) + a) \\ - \lambda(g(w - c_1 - y) + i(f(y) - y) - (1+r)(w - c_1 - y) - a).$$

Changes in z and a affect private consumption and investment through their effects on the arguments of the private sector's demand functions. These arguments are r , a , t and w , of which t and w are given. In addition to the direct effect of changes in a , we also have to take account of the indirect effects of changes in z and a through the equilibrium rate of interest. Differentiating with respect to z we obtain

$$U_1 \frac{\partial c_1}{\partial r} \frac{\partial r}{\partial z} + U_2 \left\{ f'(y) - i(f'(y) - 1) \right\} \frac{\partial y}{\partial r} \frac{\partial r}{\partial z} \\ - \left(\frac{\partial c_1}{\partial r} + \frac{\partial y}{\partial r} \right) \frac{\partial r}{\partial z} (1+r) + z \frac{\partial r}{\partial z} \\ - \lambda \left\{ -g'(z) \left(\frac{\partial c_1}{\partial r} + \frac{\partial y}{\partial r} \right) \frac{\partial r}{\partial z} + i(f'(y) - 1) \frac{\partial y}{\partial r} \frac{\partial r}{\partial z} \right. \\ \left. - z \frac{\partial r}{\partial z} + (1+r) \left(\frac{\partial c_1}{\partial r} + \frac{\partial y}{\partial r} \right) \frac{\partial r}{\partial z} \right\} = 0.$$

Here $\partial r/\partial z$ can be divided out. We can also substitute from the first-order conditions for the private sector, (8) and (9); several terms then cancel out and we are left with

$$(17) \quad U_2 - \lambda \left\{ -g'(z) \left(\frac{\partial c_1}{\partial r} + \frac{\partial y}{\partial r} \right) + \left(1 + \frac{r}{1-i} \right) \frac{\partial y}{\partial r} \right. \\ \left. + (1+r) \frac{\partial c_1}{\partial r} - z \right\} = 0.$$

Differentiating with respect to a yields¹

$$(U_1 - (1+r)U_2) \frac{\partial c_1}{\partial a} + U_2 \\ - \lambda \{ (-g'(z) + (1+r)) \frac{\partial c_1}{\partial a} - 1 \} = 0.$$

The first term on the left vanishes because of (8), and we are then left with

$$(18) \quad U_2 - \lambda \{ (-g'(z) + (1+r)) \frac{\partial c_1}{\partial a} - 1 \} = 0.$$

Substituting from (18) into (17) we can eliminate λ ; solving for $g'(z)$ we then obtain

$$g'(z) = \left[(1+r) \left(\frac{\partial c_1}{\partial r} - z \frac{\partial c_1}{\partial a} \right) + \left(\frac{r}{1-i} \right) \frac{\partial y}{\partial r} \right] / \left[\left(\frac{\partial c_1}{\partial r} - z \frac{\partial c_1}{\partial a} \right) + \frac{\partial y}{\partial r} \right].$$

Using (15) and (16) we have that

$$\frac{\partial c_1}{\partial r} - z \frac{\partial c_1}{\partial a} = \frac{1}{1+r} (w - c_1 - y - z) \frac{\partial c_1}{\partial w} + \frac{1}{D} U_2.$$

Using (1), the first term on the right vanishes, so that we have

$$(19) \quad \frac{\partial c_1}{\partial r} - z \frac{\partial c_1}{\partial a} = \frac{1}{D} U_2 = \left(\frac{\partial c_1}{\partial r} \right) U = \text{constant} \cdot \left(\frac{\partial c_1}{\partial r} \right) U,$$

which is the substitution effect or the compensated interest derivative of consumption. Our solution then becomes

$$(20) \quad g'(z) = \frac{(1+r) \left(\frac{\partial c_1}{\partial r} \right) U + \left(1 + \frac{r}{1-i} \right) \frac{\partial y}{\partial r}}{\left(\frac{\partial c_1}{\partial r} \right) U + \frac{\partial y}{\partial r}} - 1 + \rho.$$

Thus, we can conclude that the public sector's discount rate should be a weighted average of the rate facing consumers and the tax-distorted rate used by firms, the weights being the compensated interest derivative of consumption and the interest derivative of private investment, respectively. Since the weights are of the same sign, we can also conclude that the optimal discount rate is intermediate between the two private rates, i.e.

$$(21) \quad 1+r < 1+\rho < 1+r/(1-i).$$

The exact determination of the optimal rate requires empirical estimates of the two interest derivatives.

In the initial period, the public sector bids resources away from the private sector by borrowing, thereby changing the rate of interest. The

¹ We have suppressed the term involving $\partial r/\partial a$ for the following reason. The term showing the indirect effect on c_1 of a through r can be written as $(\partial L/\partial r) \cdot (\partial r/\partial a)$. But from the derivation of (17) we know that $\partial L/\partial r$ is in fact zero, and so this term vanishes. This holds true for all the models considered in this article, including the π -consumer model in the Appendix.

Interest derivatives in (20) are those of demand schedules for present resources, and the discount rate determined by this formula may thus be interpreted in terms of opportunity cost. Indeed, (20) can be said to give the marginal opportunity cost of funds. Since $1+r$ measures the marginal opportunity cost of transferring a unit of resources from private consumption, and since $1+r/(1-t)$ is the measure for transfers from private investment, a unit of resources transferred from the private to the public sector should be valued according to how much of it comes out of consumption and how much out of investment. This interpretation accords well with that of Ramsey [6].

However, this interpretation raises the question why the weighted average formula contains the compensated interest derivative of consumption and not the gross derivative, including the income effect. Clearly, it is the transfer payment which leads to this result. Looking at the government's budget constraint (11), we see that for given z and y , there must be a negative relationship between b and a ; the more the government borrows, the less it pays in transfers. Since the income effect applies to the amount lent to the government, and since a change in that amount is always accompanied by an opposite change in the amount of the transfer, the income effect cancels out. The existence of lump-sum transfers in a world of distortive taxes is perhaps something of a paradox. But it must be kept in mind that the existence of a non-distortive redistribution scheme lies as an implicit assumption in all policy recommendations that are based on economic efficiency alone. Moreover, it seems to be a fact that distortive taxes and—at least approximately—lump-sum transfers do co-exist in the real world.

V. FOREIGN BORROWING

For many countries a discussion of public investment which does not consider foreign borrowing as a source of funds is seriously incomplete. We shall accordingly include in our discussion a model of an open economy in which the bonds issued by the government are also bought by foreign investors. The supply of foreign lending is assumed to depend positively¹ on the rate of interest. Thus, letting l be total lending by foreigners, we have that

$$(22) \quad l = k(r), \quad f'(r) > 0.$$

A word of explanation is in order concerning the treatment of the foreign sector of the model. There is no explicit mention of exports and imports and of the terms of trade. There is no difficulty, however, in introducing terms-of-trade effects in our model; this can be done either explicitly or implicitly via the marginal cost of foreign borrowing. Inverting (22) into $r = r(l)$, that marginal cost is $r + r'(l)$, and one can

define $r'(l)$ so as to include terms-of-trade effects. We also remark that since we have assumed full employment of domestic resources, we must think of foreign borrowing as borrowing of real resources.

The aggregate constraint on resources in the first period now becomes

$$(23) \quad c_1 + y + z = w + l.$$

Consumers' budget constraint remains as in the previous section; what is changed is the budget constraint of the government, which is now

$$(24) \quad g(z) + t(U(y) - y) = (1+r)(b+k(r)) + a.$$

From (23) and the first-period budget constraint for consumers we have that

$$(25) \quad z = k(r) + b = k(r) + w - c_1 - y.$$

Substitution from (25) into (24) yields

$$(26) \quad g(k(r) + w - c_1 - y) + t(U(y) - y) = (1+r)k(r) + w - c_1 - y + a.$$

Our Lagrangian function is now changed to read

$$\begin{aligned} L = & U(c_1, f(y) - t(U(y) - y)) + (w - c_1 - y)(1+r) + a \\ & - \lambda(g(k(r) + w - c_1 - y) + t(U(y) - y) \\ & - (1+r)k(r) + w - c_1 - y) - a). \end{aligned}$$

Differentiating with respect to z yields

$$\begin{aligned} U_1 \frac{\partial c_1}{\partial r} \frac{\partial r}{\partial z} + U_2 \left\{ (f'(y)(1-t) + t) \frac{\partial y}{\partial r} \frac{\partial r}{\partial z} \right. \\ \left. + (w - c_1 - y) \frac{\partial r}{\partial z} - (1+r) \left(\frac{\partial c_1}{\partial r} + \frac{\partial y}{\partial r} \right) \frac{\partial r}{\partial z} \right\} \\ - \lambda \left\{ g'(z) \left(f'(r) - \frac{\partial c_1}{\partial r} - \frac{\partial y}{\partial r} \right) \frac{\partial r}{\partial z} + t \left(f'(y) - 1 \right) \frac{\partial y}{\partial r} \frac{\partial r}{\partial z} \right. \\ \left. - (1+r) \left(f'(r) - \frac{\partial c_1}{\partial r} - \frac{\partial y}{\partial r} \right) \frac{\partial r}{\partial z} - z \frac{\partial r}{\partial z} \right\} = 0. \end{aligned}$$

Dividing out $\partial r/\partial z$ and using (8) and (9) in cancelling terms, we can rewrite this condition as

$$(27) \quad U_2(w - c_1 - y) - \lambda \left\{ g'(z) \left(f'(r) - \frac{\partial c_1}{\partial r} - \frac{\partial y}{\partial r} \right) \right. \\ \left. + \left(1 + \frac{r}{1-t} \right) \frac{\partial y}{\partial r} + (1+r) \frac{\partial c_1}{\partial r} - (1+r) \left(f'(r) - z \right) \right\} = 0.$$

Differentiating with respect to a we obtain

$$(28) \quad U_2 - \lambda((-g'(z) + (1+r))(\partial c_1/\partial a) - 1) = 0.$$

¹ The presence of income effects abroad might conceivably make the derivative negative, but we shall not pay any attention to this possibility.

Eliminating λ between the two equations we can then solve for $g'(r)$:

$$g'(z) = \frac{(1+r) \left(\frac{\partial c_1}{\partial r} - (w - c_1 - y) \frac{\partial c_2}{\partial a} \right) + \left(1 + \frac{r}{1-i} \right) \frac{\partial y}{\partial r} - (1+r)l'(r) - k(r)}{\left(\frac{\partial c_1}{\partial r} - (w - c_1 - y) \frac{\partial c_2}{\partial a} \right) + \frac{\partial y}{\partial r} - l'(r)}$$

Using (19) and defining

$$(29) \quad \epsilon = l'(r)/l(r)$$

as the elasticity of the foreign supply curve for loans,³ we can rewrite the solution as

$$(30) \quad g'(z) = \frac{(1+r) \left(\frac{\partial c_1}{\partial r} \right) U + \left(1 + \frac{r}{1-i} \right) \frac{\partial y}{\partial r} - \left[1 + r \left(1 + \frac{1}{i} \right) \right] l'(r)}{\left(\frac{\partial c_1}{\partial r} \right) U + \frac{\partial y}{\partial r} - l'(r)} - 1 + r.$$

This is another formula of the weighted average type. Besides the discount rates used by consumers and firms, the elasticity-adjusted rate on foreign loans now also enters the formula, weighted by the interest derivative of the foreign loans supply function. All three weights are of the same sign, and the public sector's discount rate will then be intermediate between $(1+r)$ and $[1+r/(1-i)]$ or $[1+r(1+1/i)]$, whichever is the higher. To determine the optimal discount rate numerically, we need empirical estimates of the parameters $\partial c_1/\partial r$, $\partial y/\partial r$ and $l'(r)$.

One special feature of (30) deserves to be noted. Contrary to what is the case in the solution (20) for the closed economy, the solution does not reduce to the "first best" $g'(z) = 1+r$ when $i=0$. The reason is that the positive elasticity of the foreign loan supply curve calls for deviation from the "internal" optimality conditions, as recognized by the optimal tariff arguments in the international trade literature [4]. For $i=0$ the optimal discount rate is a weighted average of the domestic rate $1+r$ and the marginal rate on foreign loans, the weights being in fact the interest derivatives of the domestic and foreign supply schedules for government loans.

VI. FOREIGN DIRECT INVESTMENT

In Section V we introduced international capital movements, in the form of foreign borrowing and lending. There remains to consider

³ Note that the denominator of ϵ is total foreign borrowings by the country under consideration, not total gross borrowings of all countries combined. It might therefore be more accurate to refer to ϵ as "the elasticity of the supply curve for international loans divided by the country's share in international loans".

international mobility of investment itself, i.e. direct foreign investment. We assume that this investment is partly in the private, partly in the public sector of the economy. The public investment (ζ), which results in a future output of $k(\zeta)$, consists of projects that will be undertaken irrespective of the country's own economic policies, but not necessarily in the country itself. Thus as an example we might think of a harbour which will be built somewhere on the North Sea coast and which is certain to be constructed either in country A or in country B ; which interest rate should country A use to evaluate the desirability—for its own citizens—of having this investment undertaken here rather than in B ? Private investment by foreign firms (x), resulting in a future output of $k(x)$, will be assumed to be subject to income taxation at the rate θ on the same basis as domestic firms. What is the optimal tax rate on foreign profits? And how does the presence of foreign investment affect the discount rate to be used for public domestic investment? These are the three questions to which we address ourselves.

We assume that foreign private investment is also equity-financed and governed by maximization of present value. It is then clear that total investment by foreign firms is determined by the condition

$$(31) \quad k'(x) = 1 + r/(1-\theta), \quad k''(x) < 0.$$

The volume of foreign lending for domestic public investment is, as in the previous section,

$$(32) \quad z - b = z + c_1 + y - w = l.$$

The last equation is the constraint on initial resources. Note that ζ does not enter into this constraint; this is because foreign loans for public investment of that type is tied to the investment project, which is financed in its entirety by foreign funds.

In the previous section we took the volume of foreign lending to be a function of the rate of interest. It may be instructive to start here with the inverse relationship between the rate of interest and the volume of lending, viz.

$$(33) \quad r = r(l), \quad r'(l) > 0.$$

Note that ζ does not enter this function. The reason is that, as far as its effect on the interest rate is concerned, ζ should be treated as given, since it will be undertaken anyway, either here or abroad.⁴ However, since interest has to be paid on this amount too, the total interest bill to be paid in the second period is $r(l)(b+l+\zeta)$. We can now invert (33) into our original relationship $l = k(r)$ with $l'(r) > 0$, so that the interest bill becomes $r(b+k(r)+\zeta)$.

With all this the government's budget equation becomes

$$(34) \quad g(z) + k(\zeta) + r(l)(y) + \theta k(x) - x = (1+r)(b+k(r)+\zeta) + a.$$

⁴ This amounts to assuming that the world total of internationally mobile public investment is insensitive to r , or that any such sensitivity is of the second order of smallness.

This says as before that sales plus tax revenue must equal principal plus interest on loans, any surplus being paid out as a transfer.

Again, the private sector's budget constraint is the same as before (7), and so are the first-order conditions for the private sector (8)-(9). Our Lagrangian function now becomes

$$(35) \quad L = U(c_1, f(y) - f(f(y) - y) + (w - c_1 - y)(1 + r) + a) \\ - \lambda(g(k(r) + w - c_1 - y) + h(\zeta) + f(f(y) - y) \\ + \theta(k(x) - x) - (1 + r)(k(r) + w - c_1 - y) + \zeta - a).$$

Here we have substituted from (32) in the government budget constraint (34).

We now wish to determine the optimum amount to acquire of internationally mobile public investment, the optimal tax rate on the profits of foreign firms, and the optimal level of domestic public investment, or, equivalently, the optimal discount rate for domestic public investment. We therefore differentiate with respect to ζ , θ , x and a to obtain our first-order maximum conditions. The first of these conditions is

$$(36) \quad h'(\zeta) = 1 + r, \quad h'(\zeta) < 0.$$

The simplicity of this rule results from the assumption that internationally mobile public investment will be undertaken abroad if it is not undertaken here, and so the effect of such investment on the interest rate becomes irrelevant.

Differentiation with respect to θ yields

$$(37) \quad \frac{\partial(k(x) - x)}{\partial\theta} \frac{\theta}{k(x) - x} = -1.$$

The tax rate θ should be such that the elasticity of foreign investment income with respect to the tax rate is equal to minus one; this is simply a rule for maximizing tax revenue.¹

Differentiating with respect to x and using the first-order conditions (8) and (9), we obtain

$$(38) \quad U_2(w - c_1 - y) - \lambda \left\{ g'(z) \left(f(r) - \frac{\partial c_1}{\partial r} - \frac{\partial y}{\partial r} \right) \right. \\ \left. + (1 + r) \frac{\partial c_1}{\partial r} + \left(1 + \frac{r}{1 - i} \right) \frac{\partial y}{\partial r} - \left[1 + r \left(1 + \frac{1}{\epsilon} \cdot \frac{k(r) + \zeta}{k(r)} \right) \right] f'(r) \right. \\ \left. + \theta(k'(x) - 1) \frac{\partial x}{\partial r} - (w - c_1 - y) \right\} = 0.$$

Differentiation with respect to a yields

$$(39) \quad U_3 - \lambda((-g'(z) + (1 + r)(\partial c_1 / \partial a) - 1)) = 0.$$

We can now eliminate λ between the two equations and solve for

¹ If we think of countries as competing for internationally mobile investment by their taxation rules, the situation should be considered as a game-theoretic one. In (37) the policies of other countries are taken as given.

$g'(z)$. The solution is

$$g'(z) = \frac{(1 + r) \left(\frac{\partial c_1}{\partial r} \right) U + \left(1 + \frac{r}{1 - i} \right) \frac{\partial y}{\partial r} - \left[1 + r \left(1 + \frac{1}{\epsilon} \cdot \frac{k(r) + \zeta}{k(r)} \right) \right] f'(r)}{\left(\frac{\partial c_1}{\partial r} \right) U + \frac{\partial y}{\partial r} - f'(r)} \\ + \frac{\theta(k'(x) - 1) \frac{\partial x}{\partial r}}{\left(\frac{\partial c_1}{\partial r} \right) U + \frac{\partial y}{\partial r} - f'(r)}.$$

The first term on the right is identical to our previous solution (30) except for the factor $(k(r) + \zeta)/k(r)$ which "corrects" the elasticity ϵ for the fact that interest has to be paid on funds for public investment financed by foreign loans. The second term is obviously due to the existence of profits on private foreign investment. To interpret this term we note that total tax revenue from this source is equal to $R = \theta(k(x) - x)$. From (31) we know that private foreign investment will be curtailed with a rise in the rate of interest. We can then calculate the marginal tax revenue (with given tax rate) from a change in the interest rate as

$$(41) \quad \frac{\partial R}{\partial r} \theta = \theta(k'(x) - 1) \frac{\partial x}{\partial r} = \frac{\theta r}{1 - \theta} \frac{\partial x}{\partial r}$$

and we know that this expression is negative. We now take the differential of the second equation in (32):

$$dz = -(\partial c_1 / \partial r) dr - (\partial y / \partial r) dr + f'(r) dr.$$

This can be re-written as $-dz/dr = \partial c_1 / \partial r + \partial y / \partial r - f'(r)$. Since we know that the optimal solution involves a simultaneous determination of z and a so as to compensate consumers for loss of present consumption, we can write

$$(42) \quad -\frac{dz}{dr} = -\left(\frac{\partial c_1}{\partial r} \right) U + \frac{\partial y}{\partial r} - f'(r).$$

This expression is negative, as we know. We can now put together (41) and (42) to interpret the second term (in 40) as

$$-\frac{\partial R}{\partial r} \theta \frac{dr}{dz} = -\frac{\partial R}{\partial r} \theta.$$

This term is accordingly to be interpreted as the loss of tax revenue from foreign private investment following an increase in domestic public investment which drives up the rate of interest. This is in the nature of an "externality" which must be taken into account in determining the optimal level of public investment. Eq. (40) can then be

written as

$$(43) \quad g'(z) = \frac{(1+r)\left(\frac{\partial c_1}{\partial r}\right)U + \left(1 + \frac{r}{1-i}\right)\frac{\partial y}{\partial r} - \left[1 + r\left(1 + \frac{1}{e} \frac{K(r)}{r}\right)\right]r(r)}{\left(\frac{\partial c_1}{\partial r}\right)U + \frac{\partial y}{\partial r} - r(r)}$$

$$= \frac{\frac{\partial R}{\partial r} \theta - 1 + p}{\text{def}}$$

This says that the marginal loss of tax revenue should be counted as a part of the opportunity cost of public investment. Alternatively, we might subtract this loss from the marginal product of capital instead of adding it to the opportunity cost; the interpretation would then be that because of the "externality" from loss of tax revenue the marginal product of public investment should be taken as lower than implied by its production function alone.¹

VII. SOME NUMERICAL EXAMPLES

It may be instructive to examine a simple numerical example to see the actual workings of the model. We start with a closed economy with the following properties:

$$r = 0.05, \quad i = 0.50,$$

$$(\partial c_1 / \partial r)X(r/c_1)U = -0.10$$

$$(\partial y / \partial r)X(r/y) = -0.50,$$

$$c_1 = 100, \quad y = 40.$$

The assumptions about elasticities reflect the common belief that investment is more interest elastic than consumption.

From these values we easily compute the relevant derivatives as: $\partial c_1 / \partial r = -200$, and $\partial y / \partial r = -400$.

We then use (20) to compute the public discount rate:

$$\left[-(1 + 0.05)200 - \left(1 + \frac{0.05}{0.5}\right)400 \right] / (-400 - 200) = 1 + p \approx 1.083.$$

Thus, the public sector's discount rate should be 8.3 per cent., as compared to a rate of 5 per cent. for consumers and 10 per cent. for private firms.

Let us now take the model of Section V with foreign borrowing but no direct investment. In addition to our previous numerical values we assume that

¹ In general, of course, externalities of various kinds may be quite important in public investment decisions, and corrections of the same general nature would then be appropriate.

$$e = 0.5 \quad \text{and} \quad K(r) = 10,$$

implying

$$r(r) = \frac{0.5 \cdot 10}{0.05} = 100.$$

We then get from (30):

$$\frac{-(1 + 0.05)200 - \left(1 + \frac{0.05}{0.5}\right)400 - \left[1 + 0.05\left(1 + \frac{1}{0.5}\right)\right]100}{-400 - 200 - 100}$$

$$= 1 + p \approx 1.093.$$

Thus, the existence of an imperfectly elastic supply of foreign loans raises the public discount rate to 9.3 per cent.

Finally, we take the model of Section VI and assume that (36) implies $\zeta = 2$ and that (37) implies $\theta = 0.4$, so that $\frac{r}{1-\theta} = \frac{0.05}{0.6} = 0.083$. We further assume:

$$(\partial x / \partial r)X(r/x) = -0.7, \quad \text{and} \quad x = 15.$$

We then get from (40):

$$\frac{-1.05 \cdot 200 - 1 \cdot 10 \cdot 400 - \left[1 + 0.05\left(1 + \frac{1 \cdot 10 + 2}{0.5 \cdot 10}\right)\right]100}{-200 - 400 - 100}$$

$$+ \frac{0.4 \cdot \frac{0.05}{0.6} (-210)}{-200 - 400 - 100} \approx 1.10 + 0.01 = 1.11.$$

p is now higher than the rate used by private firms. This is partly the result of the government paying interest on foreign funds for public investment (ζ), and partly of taking account of the tax revenue lost by the reduction of private foreign investment as a result of an increase in the interest rate.

Summing up the numerical illustration of Section VI, we have four rates of return in our hypothetical economy:

5 per cent. is the rate earned by consumers on their bond holdings, and hence the marginal rate of substitution between present and future consumption.

5 per cent. is also the rate to be used in evaluating the attractiveness of internationally-mobile public investment projects, like building facilities for world fairs or Olympic games, providing offices for international organizations, etc.

8.3 per cent. is the rate used by foreign private firms subject to a corporate profits tax at the reduced rate of 40 per cent; it is the marginal productivity of foreign private capital, like oil refineries built

by subsidiaries of foreign corporations. 10 per cent. is the rate used by domestic private firms subject to a 50 per cent. corporate profits tax; hence it is the marginal productivity of domestic private capital, like oil refineries built by domestic corporations.

11 per cent is the rate to be used for domestic public investment projects, like building facilities for schools or universities, providing offices for national or local administration, etc.¹

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APPENDIX

The *n*-consumer Case

To demonstrate that the Pareto-optimal solution for the *n*-consumer case is the same as the one we have arrived at in the text, we shall rework our model in Sections II-IV under the assumption of *n* consumers and *n* firms. The assumption of many firms is not really very interesting and we make it only for completeness. For notational convenience we assume that there is one firm for each consumer; one might easily amend this so as to allow each consumer to own shares in many firms.

The aggregate resources constraints for the economy are:

(A.1) $c_1 + y + z = w,$

(A.2) $c_s = \sum_i f_i(y_i) + g(z).$

Here f_i is the production function of individual *i*. His utility function is

(A.3) $U^i = U^i(c_{1i}, c_{2i}), \quad i = 1, \dots, n.$

By definition we also have

(A.4) $\sum_i c_{1i} = c_1, \quad \sum_i c_{2i} = c_2, \quad \sum_i y_i = y.$

Consumer *i*'s budget constraint in the first period is

(A.5) $c_{1i} + y_i + b_i = w_i,$

where b_i is the amount he lends to the government, and w_i is his initial wealth. His future consumption is

(A.6) $c_{2i} = f_i(y_i) - (U_i(y_i) - \gamma_i) + b_i(1+r) + a_i.$

Here a_i is the lump-sum transfer received from the government in the second period; it can be positive or negative.

¹ It should be repeated that these figures are purely illustrative and have no particular claim at realism.

Substituting from (A.5) into (A.6) and then into the utility function, we have that

(A.7) $U^i = U^i(c_{1i}, f_i(y_i) - (U_i(y_i) - \gamma_i) + (w_i - c_{1i} - \gamma_i)(1+r) + a_i).$

Maximizing with respect to c_{1i} and γ_i , we obtain the first-order conditions

(A.8) $U_i^i / U_1^i = 1+r, \quad i = 1, \dots, n.$

(A.9) $f_i'(y_i) = 1+r(1-r), \quad i = 1, \dots, n.$

To make the private budget constraints consistent with full utilization of resources we must assume

(A.10) $\sum_i b_i = z.$

The constraint on the public sector in the second period is that the sum of transfer of payments be equal to the surplus in public production, i.e.

(A.11) $g(z) + r \sum_i (U_i(y_i) - \gamma_i) - (1+r) \sum_i b_i = \sum_i a_i,$

or, after substitution from (A.10) and (A.5),

(A.12) $g(\sum_i (w_i - c_{1i} - \gamma_i)) + r \sum_i (U_i(y_i) - \gamma_i) - (1+r) \sum_i (w_i - c_{1i} - \gamma_i) = \sum_i a_i.$

We can now find the conditions for our constrained Pareto optimum by maximizing a weighted sum of individual utilities, using arbitrary positive weights β_i , subject to (A.12). Writing U^i as $\ln(U_i(y_i) - \gamma_i)$ Lagrangian

(A.13) $L = \sum_i \beta_i U^i - \lambda [g(\sum_i (w_i - c_{1i} - \gamma_i)) + r \sum_i (U_i(y_i) - \gamma_i) - (1+r) \sum_i (w_i - c_{1i} - \gamma_i) - \sum_i a_i].$

This expression is to be maximized with respect to z and the a_i , which are the control variables of the government. The maximum condition for a_i is

(A.14) $\beta_i U_1^i - \lambda [(-g'(z) + (1+r)(\partial c_{1i} / \partial a_i) - 1)] = 0, \quad i = 1, \dots, n,$

and for z it is

(A.15) $\sum_i \beta_i \left\{ U_1^i \frac{\partial c_{1i}}{\partial z} + U_1^i \left[f_i'(y_i) \frac{\partial y_i}{\partial z} - (U_i(y_i) - \gamma_i) \frac{\partial \gamma_i}{\partial z} + (w_i - c_{1i} - \gamma_i) \frac{\partial r}{\partial z} - \left(\frac{\partial c_{1i}}{\partial r} + \frac{\partial \gamma_i}{\partial r} \right) \frac{\partial r}{\partial z} (1+r) \right] \right\} - \lambda \left\{ g'(z) \sum_i \left(-\frac{\partial c_{1i}}{\partial r} - \frac{\partial \gamma_i}{\partial r} \right) \frac{\partial r}{\partial z} + r \sum_i (U_i(y_i) - \gamma_i) \frac{\partial \gamma_i}{\partial r} \frac{\partial r}{\partial z} - \sum_i (w_i - c_{1i} - \gamma_i) \frac{\partial r}{\partial z} - (1+r) \sum_i \left(-\frac{\partial c_{1i}}{\partial r} - \frac{\partial \gamma_i}{\partial r} \right) \frac{\partial r}{\partial z} \right\} = 0.$

$\partial r / \partial z$ can be cancelled out from all terms in this expression. We can then eliminate β_i from (A.14); λ is then seen to cancel out also. By further rearranging and cancelling of terms we obtain

$$g'(z) = \left[(1+r) \sum_i \left(\frac{\partial c_{1i}}{\partial r} \right) U_1^i + \left(1 + \frac{r}{1-r} \right) \sum_i \frac{\partial \gamma_i}{\partial r} \right] / \left[\sum_i \left(\frac{\partial c_{1i}}{\partial r} \right) U_1^i + \sum_i \frac{\partial \gamma_i}{\partial r} \right].$$

Using (A.4) we can rewrite this as

$$(A.16) \quad f'(z) = \frac{(1+r) \left(\frac{\partial C_1}{\partial r} \right)_{\text{all } U^i} + \left(1 + \frac{r}{1-t} \right) \frac{\partial Y}{\partial r}}{\left(\frac{\partial C_1}{\partial r} \right)_{\text{all } U^i} + \frac{\partial Y}{\partial r}} \stackrel{\text{def}}{=} 1+r.$$

This is exactly the same expression as in (20) in the main text.

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